EXTRA EXERCISE FOR WEEK 5: SOLUTIONS

1. Using Taylor's formula, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!}x^{n+1}, \quad (0 < \theta < 1)\,.$$

Take $x = 1$, $e = 1 + 1 + \frac{1}{2!} + ... + \frac{1}{n!} + \frac{e^\theta}{(n+1)!}$
Since $0 < \theta < 1$, thus $0 < e^\theta < e < 3$, and $\frac{e^\theta}{(n+1)!} < \frac{3}{(n+1)!}$
Let $\frac{3}{(n+1)!} \le 10^{-6}$,, then $(n+1)! \ge 10^{-6}$,
Since 10!=3628800, 9!=362880 the smallest $n = 9$ such that the above inequality holds.
Therefore $e \approx 1 + 1 + \frac{1}{2!} + ... + \frac{1}{9!} \approx 2.718282$.

2. The stationary points are those points (x, y) for which

$$\begin{cases} \dfrac{\partial f}{\partial x} = -2\sin 2x \cdot \sin y = 0 \Rightarrow \sin 2x = 0 \text{ or } \sin y = 0 \Rightarrow x = \dfrac{n\pi}{2} \text{ or } y = m\pi, \\ \dfrac{\partial f}{\partial y} = \cos 2x \cos y = 0 \Rightarrow \cos 2x = 0 \text{ or } \cos y = 0 \Rightarrow x = \dfrac{(2n+1)\pi}{4} \text{ or } y = m\pi + \dfrac{\pi}{2}, \\ \dfrac{\partial f}{\partial z} = 2z = 0 \Rightarrow z = 0. \end{cases}$$

Thus the stationary points of f are $\left(\frac{n\pi}{2}, m\pi + \frac{\pi}{2}, 0\right)$ and $\left(\frac{(2n+1)\pi}{4}, m\pi, 0\right)$ where $m, n \in \mathbb{Z}$.
The matrix of the Hessian at (x, y, z) is

$$\mathrm{Hess}(f)(x,y,z) = \begin{pmatrix} -4\cos 2x \sin y & -2\sin 2x \cos y & 0 \\ -2\sin 2x \cos y & -\cos 2x \sin y & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

At $\left(\frac{n\pi}{2}, m\pi + \frac{\pi}{2}, 0\right)$, the matrix of the Hessian is

$$\begin{pmatrix} -4\cos n\pi \sin((2m+1)\pi/2) & 0 & 0 \\ 0 & -\cos n\pi \sin((2m+1)\pi/2) & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let λ_1, λ_2, λ_3 denote the eigenvalues of the above Hessian matrix.

$$\lambda_1 = -4\cos n\pi \sin((2m+1)\pi/2) = \begin{cases} -4 & \text{if } n+m \text{ even}, \\ 4 & \text{if } n+m \text{ odd}, \end{cases}$$

$$\lambda_2 = -\cos n\pi \sin((2m+1)\pi/2) = \begin{cases} -1 & \text{if } n+m \text{ even,} \\ 1 & \text{if } n+m \text{ odd,} \end{cases}$$

and $\lambda_3 = 2$. Thus f has local minima at $\left(\frac{n\pi}{2}, \frac{(2m+1)\pi}{2}, 0\right)$ if $n+m$ is odd. Otherwise, $\left(\frac{n\pi}{2}, \frac{(2m+1)\pi}{2}, 0\right)$ are saddle points.

At $\left(\frac{(2n+1)\pi}{4}, m\pi, 0\right)$, the matrix of the Hessian is

$$\begin{pmatrix} 0 & -2\sin((2n+1)\pi/2)\cos m\pi & 0 \\ -2\sin((2n+1)\pi/2)\cos m\pi & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$ denote the eigenvalues of the above Hessian matrix. Then $\hat{\lambda}_1+\hat{\lambda}_2+\hat{\lambda}_3 = 2$, $\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3 = -8$ Hence f has saddle points at $\left(\frac{(2n+1)\pi}{4}, m\pi, 0\right)$.

3. Applying the chain rule, we have

$$\begin{aligned} \frac{\partial z}{\partial x} - \left(e^z + xe^z \frac{\partial z}{\partial x}\right) &= 0 \\ \Longrightarrow \frac{\partial z}{\partial x} &= \frac{e^z}{1-xe^z} \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial^2 z}{\partial x} &= \frac{e^z \frac{\partial z}{\partial x}(1-xe^z) - e^z\left(-e^z - xe^z\frac{\partial z}{\partial x}\right)}{(1-xe^z)^2} \\ &= \frac{e^z \cdot \frac{\partial z}{\partial x}}{1-xe^z} + \frac{e^{2z}\left(1+x\frac{\partial z}{\partial x}\right)}{(1-xe^z)^2} \end{aligned}$$

Substituting $\frac{\partial z}{\partial x} = \frac{e^x}{1-xe^z}$,

$$\begin{aligned} \frac{\partial^2 z}{\partial x} &= \frac{e^{2z}}{(1-xe^z)^2} + \frac{e^{2z}}{(1-xe^z)^2}\left(1+\frac{xe^z}{1-xe^z}\right) \\ &= \frac{e^{2z}}{(1-xe^z)^2} + \frac{e^{2z}}{(1-xe^z)^2}\frac{1}{1-xe^z} \\ &= \frac{e^{2z}(2-xe^z)}{(1-xe^z)^3} \end{aligned}$$